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On the exact computation of the density and of the quantiles of linear combinations of t and F random variables

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Abstract

The inversion formula for evaluation of the distribution of a linear combination of independent t and F random variables, respectively, is suggested. The method is applied to computing the exact confidence intervals for the common mean of several normal populations. This method is compared with the known approximate methods. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

In this paper we discuss a method for numerical evaluation of the distribution function and/or the density function of a linear combination of independent Student's t and Fisher–Snedecor's F random variables, respectively. The method is based on the inversion formula which leads to the one-dimensional numerical integration. The characteristic function of the t and the F distribution depends on the special mathematical functions possibly of complex argument. In particular, the characteristic function of the t distribution depends on the modified Bessel function of the second kind and the characteristic function of the F distribution depends on the confluent hypergeometric function of the second kind. The suggested method can be used for computing the exact quantiles of the required distributions.

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For illustration of the method we consider the problem of computing the exact confidence intervals for the common mean of k normal populations. We do not suggest new methods for constructing the exact confidence interval nor discuss the optimality properties of the exact confidence intervals. Rather than that we employ the method suggested by Fairweather (1972) and that suggested by Jordan and Krishnamoorthy (1996), respectively. The reader who is interested in the other methods for constructing the exact confidence interval for the common mean of several normal populations and their comparison should read the paper by Yu et al. (1999).

2. The inversion formula

Gil-Pelaez (1951) derived a version of the inversion formula which is useful for numerical evaluation of a general distribution function by one-dimensional numerical integration:

Theorem 1. *Let $\phi(t) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}tx}\,\mathrm{d}F(x)$ be a characteristic function of the one-dimensional distribution function $F(x)$. Then, for x being the continuous point of the distribution, the following inversion relationship holds true*:

$$F(x) = \frac{1}{2} - \frac{1}{\pi}\int_0^{\infty} \left(\frac{\mathrm{e}^{-\mathrm{i}tx}\phi(t) - \mathrm{e}^{\mathrm{i}tx}\phi(-t)}{2it}\right)\mathrm{d}t$$

$$= \frac{1}{2} - \frac{1}{\pi}\int_0^{\infty} \mathrm{Im}\left(\frac{\mathrm{e}^{-\mathrm{i}tx}\phi(t)}{t}\right)\mathrm{d}t. \tag{1}$$

Furthermore, it is easy to observe that if the distribution belongs to the continuous type then the density function is given by

$$f(x) = \frac{1}{2\pi}\int_0^{\infty} (\mathrm{e}^{\mathrm{i}tx}\phi(-t) - \mathrm{e}^{-\mathrm{i}tx}\phi(t))\,\mathrm{d}t$$

$$= \frac{1}{\pi}\int_0^{\infty} \mathrm{Re}(\mathrm{e}^{-\mathrm{i}tx}\phi(t))\,\mathrm{d}t. \tag{2}$$

The limit properties of the integrand in (1) are given by the following Lemma 1:

Lemma 1. *Let $F(x)$ be a distribution function of a random variable X with expectation $E(X)$ and its characteristic function $\phi(t)$. Then*

$$\lim_{t\to 0} \mathrm{Im}\left(\frac{\mathrm{e}^{-\mathrm{i}tx}\phi(t)}{t}\right) = E(X) - x, \quad \text{and} \quad \lim_{t\to\infty} \mathrm{Im}\left(\frac{\mathrm{e}^{-\mathrm{i}tx}\phi(t)}{t}\right) = 0. \tag{3}$$

Proof. We will show the first equality:

$$\lim_{t\to 0} \mathrm{Im}\left(\frac{\mathrm{e}^{-\mathrm{i}tx}\phi(t)}{t}\right) = \lim_{t\to 0} \frac{1}{\mathrm{i}}\left(\frac{\mathrm{e}^{-\mathrm{i}tx}\phi(t) - \mathrm{e}^{\mathrm{i}tx}\phi(-t)}{2t}\right)$$
$$= \frac{1}{\mathrm{i}}(\mathrm{e}^{-\mathrm{i}tx}\phi(t))'|_{t=0}$$
$$= \frac{1}{\mathrm{i}}((-\mathrm{i}x)\mathrm{e}^{-\mathrm{i}tx}\phi(t) + \mathrm{e}^{-\mathrm{i}tx}\phi'(t))|_{t=0}$$
$$= \frac{1}{\mathrm{i}}(\phi'(t)|_{t=0} - \mathrm{i}x) = E(X) - x. \tag{4}$$

The second equality is a direct consequence of the fact that the function $\mathrm{e}^{-\mathrm{i}tx}\phi(t)$ is bounded in modulus.

Consider now $X = \sum_{k=1}^{n} \lambda_k X_k$ a linear combination of independent random variables and let $\phi_{X_k}(t)$ denote the characteristic function of X_k, $k = 1,\dots,n$. The characteristic function of X is

$$\phi_X(t) = \phi_{X_1}(\lambda_1 t)\cdots\phi_{X_n}(\lambda_n t) \tag{5}$$

and the distribution function $F_X(x) = \Pr\{X \leqslant x\}$ is given by (1) with $\phi(t) = \phi_X(t)$. Notice that

$$\lim_{t\to 0} \mathrm{Im}\left(\frac{\mathrm{e}^{-\mathrm{i}tx}\phi_X(t)}{t}\right) = \sum_{k=1}^{n} \lambda_k E(X_k) - x, \tag{6}$$

$$\lim_{t\to \infty} \mathrm{Im}\left(\frac{\mathrm{e}^{-\mathrm{i}tx}\phi_X(t)}{t}\right) = 0. \tag{7}$$

Formula (1) is readily applicable to numerical approximation of the distribution function $F_X(x)$ using a finite range of integration $0 \leqslant t \leqslant T$, $T < \infty$. In general, a complex-valued function should be numerically evaluated. The degree of approximation depends on the error of truncation and the error of integration method.

An interesting application of the above inversion formula was given by Imhof (1961) who derived the formula to calculate the distribution of a linear combination of independent non-central chi-squared random variables $X = \sum_{k=1}^{n} \lambda_k X_k$, where $X_k \sim \chi^2_{v_k}(\delta_k^2)$, with v_k degrees of freedom and the non-centrality parameter δ_k^2. For review and discussion on numerical inversion of the characteristic function as a tool for obtaining cumulative distribution functions, see Waller et al. (1995).

3. Characteristic function of the t distribution

Theorem 2. *Let $X \sim t_v$ be a random variable that has Student's t distribution with v degrees of freedom with its probability density function given by*

$$f(x) = \frac{\Gamma(v/2 + \frac{1}{2})}{(\pi v)^{1/2}\Gamma(v/2)}\left(1 + \frac{x^2}{v}\right)^{-(v/2+1/2)}, \tag{8}$$

where $-\infty < x < \infty$. *Then the characteristic function of* $X \sim t_v$ *is*

$$\phi_v^t(t) = \frac{1}{2^{v/2-1}\Gamma(v/2)}(v^{1/2}|t|)^{v/2}K_{v/2}\{v^{1/2}|t|\}, \tag{9}$$

where $K_\alpha\{z\}$ *denotes the modified Bessel function of second kind.*

Proof. The characteristic function of $X \sim t_v$ is

$$\begin{aligned}
\phi_v^t(t) = E(\mathrm{e}^{\mathrm{i}tX}) &= \frac{\Gamma(v/2+\frac{1}{2})}{(\pi v)^{1/2}\Gamma(v/2)}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}tx}\left(1+\frac{x^2}{v}\right)^{-(v/2+1/2)} \mathrm{d}x \\
&= \frac{\Gamma(v/2+\frac{1}{2})v^{v/2}}{\pi^{1/2}\Gamma(v/2)}\int_{-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}tx}}{(v+x^2)^{v/2+1/2}}\,\mathrm{d}x \\
&= \frac{\Gamma(v/2+\frac{1}{2})v^{v/2}}{\pi^{1/2}\Gamma(v/2)}\left[\int_{-\infty}^{0} \frac{\mathrm{e}^{\mathrm{i}tx}}{(v+x^2)^{v/2+1/2}}\,\mathrm{d}x + \int_{0}^{\infty} \frac{\mathrm{e}^{\mathrm{i}tx}}{(v+x^2)^{v/2+1/2}}\,\mathrm{d}x\right] \\
&= \frac{\Gamma(v/2+\frac{1}{2})v^{v/2}}{\pi^{1/2}\Gamma(v/2)}\int_{0}^{\infty} \frac{\mathrm{e}^{-\mathrm{i}tx}+\mathrm{e}^{\mathrm{i}tx}}{(v+x^2)^{v/2+1/2}}\,\mathrm{d}x \\
&= \frac{\Gamma(v/2+\frac{1}{2})v^{v/2}}{\pi^{1/2}\Gamma(v/2)}\int_{0}^{\infty} \frac{2\cos(tx)}{(v+x^2)^{v/2+1/2}}\,\mathrm{d}x.
\end{aligned} \tag{10}$$

In accordance with (9.6.25) in Abramowitz and Stegun (1965, p. 376) we have

$$K_{v/2}\{tz\} = \frac{\Gamma(v/2+\frac{1}{2})(2z)^{v/2}}{\pi^{1/2}t^{v/2}}\int_{0}^{\infty} \frac{\cos(tx)}{(x^2+z^2)^{v/2+1/2}}\,\mathrm{d}x \tag{11}$$

for $\mathrm{Re}(v) \geqslant -1$, $t > 0$, and $|\arg z| < \frac{1}{2}\pi$. Choosing $z = v^{1/2}$ and using the fact that $\cos(tx) = \cos(-tx)$ we get the result. $\square$

Lemma 2. *The characteristic function of the random variable* $X \sim t_v$ *can be calculated by using* (9) *and the recurrence relation*

$$K_{k/2+1}\{z\} = K_{k/2-1}\{z\} + \frac{k}{z}K_{k/2}\{z\}, \tag{12}$$

where $z = v^{1/2}|t|$.

If v *is even*, $v = 2n$ *for some integer* n, *then* $k = 2, 4, \ldots, v-2$, *given* $K_0\{z\}$ *and* $K_1\{z\}$. *If* v *is odd*, $v = 2n+1$ *for some integer* n, *then* $k = 3, 5, \ldots, v-2$, *given* $K_{1/2}\{z\}$ *and* $K_{3/2}\{z\}$.

Proof. The proof is a direct consequence of recurrence relation (9.6.26) in Abramowitz and Stegun (1965, p. 376)

$$Z_{v-1}\{z\} - Z_{v+1}\{z\} = \frac{2v}{z}Z_v\{z\}, \tag{13}$$

where Z_v denotes $\mathrm{e}^{\mathrm{i}v\pi}K_v\{z\}$. $\square$

The following lemma is in accordance with the result given by Mitra (1978). For more details see Johnson et al. (1995, p. 367).

Lemma 3. *If v is odd, $v = 2n+1$ for some integer n, then the characteristic function of the random variable $X \sim t_v$ is*

$$\phi_v^t(t) = \varphi_n(t)\exp\{-v^{1/2}|t|\}, \tag{14}$$

where $\varphi_n(t)$ is given by the recurrence relation

$$\varphi_{k+1}(t) = \frac{vt^2}{(2k+1)(2k-1)}\varphi_{k-1}(t) + \varphi_k(t), \tag{15}$$

$k = 1,\dots,n-1$, *where* $\varphi_0(t) = 1$, *and* $\varphi_1(t) = 1 + v^{1/2}|t|$.

Proof. Eq. (10.2.17) in Abramowitz and Stegun (1965, p. 444) states that

$$\begin{aligned} K_{1/2}\{z\} &= \left(\frac{\pi}{2z}\right)^{1/2} \exp\{-z\}, \\ K_{3/2}\{z\} &= \left(\frac{\pi}{2z}\right)^{1/2} \exp\{-z\}(1+z^{-1}), \\ K_{5/2}\{z\} &= \left(\frac{\pi}{2z}\right)^{1/2} \exp\{-z\}(1+3z^{-1}+3z^{-2}). \end{aligned} \tag{16}$$

Moreover, if we define

$$f_k(z) = (-1)^{k+1}\left(\frac{\pi}{2z}\right)^{1/2} K_{k+1/2}\{z\}, \tag{17}$$

then according to Eq. (10.2.18) in Abramowitz and Stegun (1965, p. 444) we get

$$f_{k-1}(z) - f_{k+1}(z) = (2k+1)z^{-1} f_k(z) \tag{18}$$

for $k = 0, \pm 1, \pm 2, \dots$.

Let $z = v^{1/2}|t|$. Then from (17) we get

$$K_{v/2}(z) = K_{n+1/2}(z) = (-1)^{-(n+1)}\left(\frac{\pi}{2z}\right)^{-1/2} f_n(z). \tag{19}$$

Using (9) and (16) and after some further simple algebraical manipulations we get the result. □

Corollary 1. *In particular, the characteristic function $\phi_v^t(t)$ of the random variable $X \sim t_v$ with $v = 2n+1$ degrees of freedom where $n = 0,1,2,3$ is*

$$\begin{aligned} \phi_1^t(t) &= \exp\{-|t|\}, \\ \phi_3^t(t) &= (1+\sqrt{3}|t|)\exp\{-\sqrt{3}|t|\}, \\ \phi_5^t(t) &= \left(1+\sqrt{5}|t|+\frac{5}{3}t^2\right)\exp\{-\sqrt{5}|t|\}, \\ \phi_7^t(t) &= \left(1+\sqrt{7}|t|+\frac{14}{5}t^2+\frac{7\sqrt{7}}{15}|t|^3\right)\exp\{-\sqrt{7}|t|\}. \end{aligned} \tag{20}$$

4. Characteristic function of the F distribution

Theorem 3. *Let $X \sim F_{v_1,v_2}$ be a random variable that has central Fisher–Snedecor F distribution with v_1 and v_2 degrees of freedom with its probability density function given by*

$$f(x) = \frac{\Gamma(v_1/2+v_2/2)}{\Gamma(v_1/2)\Gamma(v_2/2)}\left(\frac{v_1}{v_2}\right)^{v_1/2} x^{v_1/2-1}\left(1+\frac{v_1}{v_2}x\right)^{-(v_1/2+v_2/2)}, \tag{21}$$

where $0 < x < \infty$. Then the characteristic function of $X \sim F_{v_1,v_2}$ is

$$\phi^F_{v_1,v_2}(t) = \frac{\Gamma(v_1/2+v_2/2)}{\Gamma(v_2/2)}\Psi\left(\frac{v_1}{2},1-\frac{v_2}{2};-\mathrm{i}t\frac{v_2}{v_1}\right), \tag{22}$$

where $\Psi(a,c;z)$ denotes the confluent hypergeometric function of the second kind defined by the integral equation

$$\Psi(a,c;z) = \frac{1}{\Gamma(a)}\int_0^\infty \mathrm{e}^{-zt}t^{a-1}(1+t)^{c-a-1}\,\mathrm{d}t. \tag{23}$$

Proof. See Phillips (1982). □

Lemma 4. *Denote $a = v_1/2$, $c = 1 - v_2/2$, and $z = (-\mathrm{i}tv_2/v_1)$. The series representation of* (22) *can be obtained from the following formulae*:

$$\Psi(a,c;z) = \frac{\Gamma(1-c)}{\Gamma(a-c+1)}\,{}_1F_1(a,c;z) + \frac{\Gamma(c-1)}{\Gamma(a)}z^{1-c}\,{}_1F_1(a-c+1,2-c;z) \tag{24}$$

for non-integral c. For $c = 1-n$ with $n = 1,2,\ldots,$

$$\begin{aligned}\Psi(a,1-n;z) &= z^n\Psi(a+n,n+1;z)\\ &= \frac{(-1)^{n-1}z^n}{n!\Gamma(a)}\left[{}_1F_1(a+n,n+1;z)\log z + \sum_{k=0}^{\infty}\frac{(a+n)_k}{(n+1)_k k!}\right.\\ &\qquad\left.\times\,\{\psi(a+n+k)-\psi(1+k)-\psi(1+n+k)\}z^k\right]\\ &\qquad + \frac{(n-1)!}{\Gamma(a+n)_k}\sum_{k=0}^{n-1}\frac{(a)_k}{(1-n)_k k!}z^k,\end{aligned} \tag{25}$$

where ${}_1F_1(a,c;z)$ denotes the confluent hypergeometric function of the first kind, $\psi(x) = \Gamma'(x)/\Gamma(x)$ is the logarithmic derivative of the gamma function, and where $(a)_n = a(a+1)(a+2)\cdots(a+n-1)$, $(a)_0 = 1$. The final term in (25) *is omitted if $n = 0$.*

Proof. See Phillips (1982). See also Eqs. (13.1.3) and (13.1.6) in Abramowitz and Stegun (1965, p. 504). □

Corollary 2. *When v_2 is odd then for small t we have the following expansion*:

$$\phi^F_{v_1,v_2}(t) = \frac{1}{A_0}\sum_{k=0}^{\infty}\left(A_k - \left(-\mathrm{i}t\frac{v_2}{v_1}\right)^{v_2/2}B_k\right), \tag{26}$$

where

$$A_0 = \frac{\sin(\pi v_2/2)}{\pi}\Gamma\left(\frac{v_1}{2}\right)\Gamma\left(\frac{v_2}{2}\right),$$

$$A_k = \frac{v_1/2+k-1}{k(k-v_2/2)}\left(-\mathrm{i}t\frac{v_2}{v_1}\right)A_{k-1},$$

$$B_0 = \frac{\Gamma(v_1/2+v_2/2)}{\Gamma(1+v_2/2)},$$

$$B_k = \frac{(v_1/2+v_2/2+k-1)}{k(k+v_2/2)}\left(-\mathrm{i}t\frac{v_2}{v_1}\right)B_{k-1}. \tag{27}$$

Proof. The proof is a direct consequence of (22), (24) and the expansion of ${}_1F_1(a,c;z)$:

$${}_1F_1(a,c;z) = 1+\frac{az}{c}+\frac{(a)_2z^2}{(c)_2 2!}+\cdots+\frac{(a)_nz^n}{(c)_n n!}+\cdots, \tag{28}$$

where $(a)_n = a(a+1)(a+2)\cdots(a+n-1)$, $(a)_0 = 1$. See Eq. (13.1.2) in Abramowitz and Stegun (1965, p. 504). □

Lemma 5. *For $|z|$ large, $z = (-\mathrm{i}tv_2/v_1)$, the characteristic function $\phi^F_{v_1,v_2}(t)$ given by* (22) *can be approximated by using*

$$\begin{aligned}&\Psi\left(\frac{v_1}{2}, 1-\frac{v_2}{2}; -\mathrm{i}t\frac{v_2}{v_1}\right)\\ &\quad= \left(-\mathrm{i}t\frac{v_2}{v_1}\right)^{-v_1/2}\left[\sum_{k=0}^{R}\frac{(v_1/2)_k(v_1/2+v_2/2)_k}{k!}\left(\mathrm{i}t\frac{v_2}{v_1}\right)^{-k}+\mathrm{O}\left(\left|-\mathrm{i}t\frac{v_2}{v_1}\right|^{-R}\right)\right],\end{aligned}$$

where

$$\begin{aligned}&\mathrm{O}\left(\left|-\mathrm{i}t\frac{v_2}{v_1}\right|^{-R}\right)\\ &\quad= \frac{(v_1/2)_R(v_1/2+v_2/2)_R}{R!}\left(\mathrm{i}t\frac{v_2}{v_1}\right)^{-R}\\ &\qquad\times\left[\frac{1}{2}+\frac{(\frac{1}{8}+\frac{1}{4}(1-v_2/2)-\frac{1}{2}v_1/2+\frac{1}{4}(-\mathrm{i}tv_2/v_1)-\frac{1}{4}R)}{(-\mathrm{i}tv_2/v_1)}+\mathrm{O}\left(\left|-\mathrm{i}t\frac{v_2}{v_1}\right|^{-2}\right)\right].\end{aligned}$$

Proof. See Eqs. (13.5.2) and (13.5.3) in Abramowitz and Stegun (1965, p. 508). □

Lemma 6. *The characteristic function of the random variable $X \sim F_{v_1,v_2}$ can be calculated by using* (22) *and the recurrence relations*

$$\Psi(k+1,c;z) = \frac{1}{k(c-k-1)}[\Psi(k-1,c;z)+(c-2k-z)\Psi(k,c;z)], \tag{29}$$

where $c = 1-v_2/2$ and $z = (-\mathrm{i}tv_2/v_1)$. If v_1 is even, $v_1 = 2n$ for some integer n, then $k = 3,4,\ldots,v_1/2-1$, given $\Psi(1,c;z)$ and $\Psi(2,c;z)$. If v_1 is odd, $v_1 = 2n+1$ for some *integer n, then* $k=\frac{3}{2},\frac{5}{2},\ldots,\nu_1/2-1$, *given* $\Psi(\frac{1}{2},c;z)$ *and* $\Psi(\frac{3}{2},c;z)$, *and*

$$\Psi(a,k-1;z)=\frac{1}{1+a-k}[(1-k-z)\Psi(a,k;z)+z\Psi(a,k+1;z)], \tag{30}$$

where $a=\nu_1/2$ *and* $z=(-\mathrm{i}t\nu_2/\nu_1)$. *If* ν_2 *is even,* $\nu_2=2n$ *for some integer n, then* $k=-1,-2,\ldots,2-\nu_2/2$, *given* $\Psi(a,0;z)$ *and* $\Psi(a,-1;z)$. *If* ν_2 *is odd,* $\nu_1=2n+1$ *for some integer n, then* $k=-\frac{1}{2},-\frac{3}{2},\ldots,2-\nu_2/2$, *given* $\Psi(a,\frac{1}{2};z)$ *and* $\Psi(a,-\frac{1}{2};z)$.

Proof. See the recurrence relations (13.4.15) and (13.4.16) in Abramowitz and Stegun (1965, p. 507). □

Lemma 7. *The following series expansions can be used to calculate* (22):

$$\begin{aligned}{}_1F_1(a,c;z)&=\Gamma\left(a+\frac{1}{2}\right)\mathrm{e}^{z/2}\left(\frac{4}{z}\right)^{a-1/2}\\&\quad\times\sum_{n=0}^{\infty}\frac{(n+a-\frac{1}{2})(2a-1)_n(2a-c)_n}{n!(c)_n(a-\frac{1}{2})}I_{n+a-1/2}\left\{\frac{z}{2}\right\},\end{aligned} \tag{31}$$

$$\begin{aligned}{}_1F_1(a,c;z)&=\Gamma\left(c-a+\frac{1}{2}\right)\mathrm{e}^{z/2}\left(\frac{4}{z}\right)^{c-a-1/2}\\&\quad\times\sum_{n=0}^{\infty}\frac{(-1)^n(n+c+a-\frac{1}{2})(2c-2a-1)_n(c-2a)_n}{n!(c)_n(c-a-\frac{1}{2})}I_{n+c+a-1/2}\left\{\frac{z}{2}\right\},\end{aligned} \tag{32}$$

where $I_\alpha\{z\}$ *denotes the modified Bessel function of the first kind.*

Proof. See Eqs. (7.8.7) and (7.8.8) in Luke (1980, p. 313). □

5. Application to computation of the exact confidence intervals for the common mean of several normal populations

Let us assume that we have $k\geqslant 2$ independent populations where the ith population follows $\mathrm{N}(\mu,\sigma_i^2)$ distribution with common mean μ and possibly unequal variances σ_i^2, $i=1,\ldots,k$. Let X_{ij}, $j=1,\ldots,n_i$ ($n_i\geqslant 2$), be a random sample from the ith population. We define $\bar{X}_i$ and S_i^2 as

$$\bar{X}_i=\frac{1}{n_i}\sum_{j=1}^{n_i}X_{ij},\quad S_i^2=\frac{1}{n_i-1}\sum_{j=1}^{n_i}(X_{ij}-\bar{X}_i)^2, \tag{33}$$

$i=1,\ldots,k$. The random variables $\bar{X}_i$ and S_i^2 are mutually independent and

$$\bar{X}_i\sim\mathrm{N}\left(\mu,\frac{\sigma_i^2}{n_i}\right),\qquad(n_i-1)S_i^2\sim\sigma_i^2\chi^2_{n_i-1},\quad i=1,\ldots,k. \tag{34}$$

From that we have

$$t_i = \frac{\sqrt{n_i}(\bar{X}_i - \mu)}{S_i} \sim t_{n_i-1}, \quad F_i = \frac{n_i(\bar{X}_i - \mu)^2}{S_i^2} \sim F_{1,n_i-1}, \tag{35}$$

$i = 1, \ldots, k.$

Fairweather (1972) suggested construction of the exact confidence interval for μ using a weighted linear combination of the Student's t_i statistics, namely

$$W_t = \sum_{i=1}^{k} u_i t_i, \quad u_i = \frac{(\mathrm{Var}(t_i))^{-1}}{\sum_{i=1}^{k} (\mathrm{Var}(t_i))^{-1}} = \frac{(n_i-3)/(n_i-1)}{\sum_{j=1}^{k}(n_j-3)/(n_j-1)}. \tag{36}$$

Note that $\mathrm{Var}(t_i)$ exists if $n_i > 3$. If $b_{\alpha/2}$ denotes the upper cut-off point of the distribution of W_t, such that for given $\alpha \in (0,1)$

$$P(|W_t| \leqslant b_{\alpha/2}) = 1 - \alpha, \tag{37}$$

then the exact (symmetric, two-sided) $100(1-\alpha)$%-confidence interval for μ is obtained as

$$\left[\frac{\sum_{i=1}^{k} \sqrt{n_i} u_i \bar{X}_i / S_i - b_{\alpha/2}}{\sum_{i=1}^{k} \sqrt{n_i} u_i / S_i}, \frac{\sum_{i=1}^{k} \sqrt{n_i} u_i \bar{X}_i / S_i + b_{\alpha/2}}{\sum_{i=1}^{k} \sqrt{n_i} u_i / S_i} \right]. \tag{38}$$

To derive the approximate cut-off point $b^*_{\alpha/2}$ Fairweather (1972) suggested to approximate the distribution of W_t by that of ct_v where v and $c > 0$ are determined by equating the second and fourth moments of ct_v to those of W_t. In particular, if we additionally assume that $n_i > 5$ for all $i = 1, \ldots, k$ then we get

$$v = 4 + \frac{1}{\sum_{i=1}^{k} u_i^2/(n_i-5)}, \quad c = \sqrt{\frac{v-2}{v\sum_{i=1}^{k}(n_i-3)/(n_i-1)}}. \tag{39}$$

Jordan and Krishnamoorthy (1996) suggested using a weighted linear combination of the F_i statistics, using weights inversely proportional to variances $\mathrm{Var}(F_i)$, namely

$$W_f = \sum_{i=1}^{k} w_i F_i, \quad w_i = \frac{[(n_i-3)^2(n_i-5)]/[(n_i-1)^2(n_i-2)]}{\sum_{i=1}^{k} [(n_j-3)^2(n_j-5)]/[(n_j-1)^2(n_j-2)]}. \tag{40}$$

Note that $\mathrm{Var}(F_i)$ exists if $n_i > 5$. If a_α denotes the cut-off point of the distribution of W_f, such that for given $\alpha \in (0,1)$

$$P(W_f \leqslant a_\alpha) = 1 - \alpha, \tag{41}$$

then the exact (symmetric, two-sided) $100(1-\alpha)$%-confidence interval for μ is obtained as

$$\left[\sum_{i=1}^{k} p_i \bar{X}_i - \Delta, \sum_{i=1}^{k} p_i \bar{X}_i + \Delta \right], \tag{42}$$

where

$$p_i = \frac{w_i n_i / S_i^2}{\sum_{j=1}^{k} w_j n_j / S_j^2} \tag{43}$$

Table 1
Percentage of albumin in plasma protein

Experiment	n_i	Mean	Variance
A	12	62.3	12.986
B	15	60.3	7.840
C	7	59.5	33.433
D	16	61.5	18.513

and

$$\Delta^2 = \frac{a_\alpha}{\sum_{i=1}^k w_i n_i / S_i^2} - \left\{ \sum_{i=1}^k p_i \bar{X}_i^2 - \left(\sum_{i=1}^k p_i \bar{X}_i \right)^2 \right\}. \tag{44}$$

To derive the approximate cut-off point a_α^* Jordan and Krishnamoorthy (1996) suggested to approximate the distribution of W_f by that of $cF_{k,v}$ where v and $c > 0$ are determined by equating the first two moments of $cF_{k,v}$ to those of W_f. In particular, if we assume that $n_i > 5$ for all $i = 1, \ldots, k$ then we get

$$v = \frac{4kM_2 - 2(k+2)M_1^2}{kM_2 - (k+2)M_1^2}, \quad c = \frac{v-2}{v} M_1, \tag{45}$$

where

$$M_1 = E(W_f) = \sum_{i=1}^k \frac{w_i(n_i - 1)}{(n_i - 3)} \tag{46}$$

and

$$M_2 = E(W_f)^2 = 3 \sum_{i=1}^k \frac{w_i^2 (n_i - 1)^2}{(n_i - 3)(n_i - 5)} + 2 \sum_{i>j} \frac{w_i w_j (n_i - 1)(n_j - 1)}{(n_i - 3)(n_j - 3)}. \tag{47}$$

To compare the exact and the approximate methods suggested for computing the cut-off points we provide two examples considered and analyzed by Jordan and Krishnamoorthy (1996) and Yu et al. (1999).

Example 1. Here we examine the data reported in Meier (1953) about the percentage of albumin in plasma protein in human subjects. The data given in Table 1 are based on four independent experiments. It is assumed that the samples are from normal populations.

We would like to combine the results of the four experiments in order to construct a $100(1-\alpha)\%$-confidence interval for the common mean μ on the significance level $\alpha = 0.05$. We have applied the techniques suggested by Fairweather (1972) and Jordan and Krishnamoorthy (1996).

The approximate upper cut-off point $b_{0.025}^*$ of the weighted linear combination $W_t = \sum_{i=1}^4 u_i t_{n_i - 1}$ of the Student's t_{n_i-1} statistics with weights

$$u = [0.2550, 0.2671, 0.2078, 0.2701],$$

can be computed according to (39) from the distribution of ct_v, where $v = 26.3984$ and $c = 0.5367$. This approximation leads to $b_{0.025}^* = 1.1024$ and the resulting confidence

Table 2
Selenium in non-fat milk powder

Method	n_i	Mean	Variance
A	8	105.00	85.711
B	12	109.75	20.748
C	14	109.50	2.729
D	8	113.25	33.640

interval estimate is $[59.8973, 62.1921]$ with the true coverage probability $P(|W_t| < b^*_{0.025}) = 0.9504$.

The exact value (rounded off to the fourth decimal place) of the upper cut-off point $b_{0.025}$ is $b_{0.025} = 1.1002$. This was computed numerically based on (1), (5) and (9). Finally, the exact 95%-confidence interval estimate for μ is

$$[59.8996, 62.1899].$$

The approximate cut-off point $a^*_{0.05}$ of the weighted linear combination $W_f = \sum_{i=1}^4 w_i F_{1,n_i-1}$ of the F_{1,n_i-1} statistics with weights

$$w = [0.2601, 0.3137, 0.0987, 0.3276],$$

can be computed according to (45) from the distribution of $cF_{4,v}$, where $v=15.6082$ and $c = 1.0548$. This approximation leads (using a linear interpolation of the critical values of the F distribution) to $a^*_{0.05} = 3.1918$ and the resulting confidence interval estimate is $[59.5621, 62.4430]$ with the true coverage probability $P(W_f < a^*_{0.05}) = 0.9503$.

The exact value of the cut-off point $a_{0.05}$ is $a_{0.05} = 3.1853$ which was computed numerically based on (1), (5) and (22) and the exact 95%-confidence interval estimate for μ is

$$[59.5640, 62.4410].$$

Example 2. Eberhardt et al. (1989) reported the data on Selenium in non-fat milk powder. The data are given in Table 2. They are based on four independent measurement methods. It is assumed that the samples are from normal populations.

The approximate upper cut-off point $b^*_{0.025}$ of the weighted linear combination $W_t = \sum_{i=1}^4 u_i t_{n_i-1}$ of the Student's t_{n_i-1} statistics with weights

$$u = [0.2309, 0.2645, 0.2736, 0.2309],$$

can be computed according to (39) from the distribution of ct_v, where $v = 22.5633$ and $c = 0.5428$. This approximation leads to $b^*_{0.025} = 1.1241$ and the resulting confidence interval estimate is $[108.5349, 110.7742]$ with the true coverage probability $P(|W_t| < b^*_{0.025}) = 0.9504$.

The exact value of the upper cut-off point $b_{0.025}$ is $b_{0.025} = 1.1221$ (computed numerically based on (1), (5) and (9)). The exact 95%-confidence interval estimate for μ is

$$[108.5369, 110.7722].$$

The approximate cut-off point $a^*_{0.05}$ of the weighted linear combination $W_f = \sum_{i=1}^4 w_i F_{1,n_i-1}$ of the F_{1,n_i-1} statistics, with weights

$$w = [0.1683, 0.3091, 0.3543, 0.1683],$$

can be computed according to (45) from the distribution of $cF_{4,v}$, where $v = 13.3466$ and $c = 1.0778$. This approximation leads to $a^*_{0.05} = 3.4014$ and the resulting confidence interval estimate is $[108.4530, 110.6680]$ with the true coverage probability $P(W_f < a^*_{0.05}) = 0.9511$.

The exact value of the cut-off point $a_{0.05}$ is $a_{0.05} = 3.3748$ (computed numerically based on (1), (5) and (22)) and the exact 95%-confidence interval estimate for μ is

$$[108.4588, 110.6621].$$

6. Concluding remarks

Although the presented methods are not new, we believe that it would be beneficial for statisticians to have them presented in one place, with examples of applications.

According to our knowledge the closed form of the characteristic function of the Student's t distribution as presented in Theorem 2 was unknown in statistical literature. The recurrence properties of the characteristic functions of t and F distributions were first discussed in Ifram (1972).

The above examples show that the approximate methods for computing the confidence intervals perform very well. However, we believe that the suggested method for computing the exact distribution of a linear combination of the Student's t random variables and/or the Fisher–Snedecor's F random variables can be useful in more complicated (unbalanced) situations. The method is quite general without any restriction on the number of the variables, the values of the coefficients and the involved degrees of freedom. The numerical implementation is easy, provided we have an efficient algorithm for numerical evaluation of the modified Bessel function of second kind and the confluent hypergeometric function of the second kind of a complex argument. For more details see e.g. Amos (1986) and Nardin et al. (1992).

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